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Quantum relaxation in open chaotic systems

Klaus M. Frahm

Laboratoire de Physique Quantique, UMR 5626 du CNRS, Université Paul Sabatier, F-31062 Toulouse Cedex 4, France

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Using the supersymmetry technique, we analytically derive the recent result of Casati, Maspero, and Shepelyansky [Phys. Rev. E **56**, R6233 (1997)] according to which the quantum dynamics of open chaotic systems follows the classical decay up to a new quantum relaxation time scale $t_q \sim \sqrt{t_c t_H}$. This scale is larger than the classical escape time t_c but still much smaller than the Heisenberg time t_H. For systems with orthogonal or unitary symmetry the quantum decay is slower than the classical one while for the symplectic case there is an intermediate regime in which the quantum decay is slightly faster. [S1063-651X(97)51012-9]

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The classical decay probability of a generic weakly open chaotic system obeys the exponential distribution $P_{\rm cl}(t) \propto e^{-t/t_c}$ where the mean escape time t_c characterizes the effective coupling to the outside. Motivated by recent experiments on mesoscopic cavities or microwave billiards there has been renewed interest in the problem of *quantum life times* [1–8]. For example, the quantum properties of ''chaotic'' maps with absorption [1,2] were investigated. Recently, also analytical results for the statistical distributions of the complex poles of the scattering matrix [3,5] or of the eigenvalues of the Wigner-Smith matrix of time delays [6] were found for the generic problem of chaotic scattering.

The problem of current relaxation in disordered metals, which is similar to the quantum decay of an initially localized wave packet inside a chaotic cavity or a disordered region, has been investigated by different analytical approaches [9–11] in the framework of the nonlinear σ model (replica or supersymmetry variant [12]). In these works it was shown that the classical exponential decay is strongly supressed by quantum effects for time scales larger than the Heisenberg time t_H giving rise to a log-normal distribution of relaxation times. Muzykantskii and Khmelnitskii [10] demonstrated that this is due to a nontrivial saddle point of the σ model. For the case of open one-dimensional geometries they obtained for $t > t_H$ the behavior $P(t) \sim \exp[-g \ln^2(t/t_H)]$ where $g = t_H/t_c \gg 1$ is the conductance (in units of e^2/h). This result has an important relation to the probability to find a ''nearly localized state'' in a normally metallic sample [10]. Also the quantum time evolution of open chaotic cavities was studied [13] giving a power law decay for $t \gg t_H/\min\{T_j\}$ where $0 \leqslant T_j \leqslant 1$ are the transmission coefficients of the barrier by which the cavity is coupled to the outside.

Recently, Casati, Maspero, and Shepelyansky [1] surprisingly found that for a quantum kicked rotator model with absorption [2] significant deviations from the classical behavior already appear at an earlier *quantum relaxation time scale* $t_q \sim \sqrt{t_c t_H} \ll t_H$. Their argument [1] is based on the complex eigenvalues of the nonunitary time evolution operator being typically distributed in a narrow ring of width $E_c = 1/t_c$ [2] inside the unit circle. ($t_c \gg 1$ is measured in units of the kick period.) Then t_q can be identified [1] as the inverse of their typical distance in the complex plane. This picture is indeed supported by numerical quantum simulations [1] for the kicked rotator.

In this work, we present analytical results for a similar model by mapping it onto the supersymmetric nonlinear σ model [12], which is possible due to recent progress of Altland and Zirnbauer [14] for this type of system. The σ-model in the zero-dimensional limit also applies to the case of a chaotic cavity coupled to external leads [15]. We clearly confirm the findings of Casati *et al.* that the new time scale t_q is indeed highly relevant for the problem and, additionally, we find at $t \sim t_q$ qualitatively different quantum effects for the three symmetry classes of random matrix theory [16] that are characterized by the index $\beta = 1$ for the orthogonal case (systems with time reversal symmetry and no spin mixing), $\beta = 2$ for the unitary case (broken time reversal symmetry), and $\beta = 4$ for the symplectic case (time reversal symmetry and strong spin mixing). This result supports the interpretation that the effect of weak localization (or antilocalization for $\beta = 4$) can also be observed in open systems with absorption.

We consider the quantum dynamics $|\psi(t+1)\rangle = S|\psi(t)\rangle$ of a generalized random phase kicked rotator model with the time evolution operator introduced in [2,1]

$$S_{l\tilde{l}} = e^{i\mu_l} \langle l | e^{-iV(\theta)} | \tilde{l} \rangle e^{i\mu_{\tilde{l}}}, \qquad (1)$$

where $l, \tilde{l}$ are the quantum numbers of ''angular momentum'' being conjugated to the angle θ. As in Refs. [2,1] the l space is finite: $-N/2 \leqslant l, \tilde{l} \leqslant N/2$ introducing effective absorption at the boundaries. We consider random phases μ_l [17] and a quite general periodic kick potential $V(\theta)$ (with a finite number of harmonics). The different symmetry classes are encoded in the symmetries of $V(\theta)$ [18], i.e., $V(\theta) = V(-\theta)$ for $\beta = 1$ and $V(\theta) = V_0(\theta) \mathbb{1}_2 + \Sigma_{\nu=1}^3 \sigma_\nu V_\nu(\theta)$ for $\beta = 4$. Here σ_ν are the Pauli matrices and $V_\nu(\theta)$ is even (or odd) for $\nu = 0$ (or $\nu = 1,2,3$). In the following, we consider the phase averaged quantity $P(t) = \langle |\langle 0 | S^t | 0 \rangle|^2 \rangle_\mu$ to describe the decay of a quantum state initially localized at the site $l = 0$. For short time scales this probability decays diffusively as $P(t) \propto 1/\sqrt{Dt}$ (with diffusion constant $D = \langle V'(\theta)^2 \rangle_\theta \gg 1$) whereas for longer time scales and large system size ($t, N \gg D$) quantum localization leads to the saturation $P(t) \propto 1/\xi$ with the localization length

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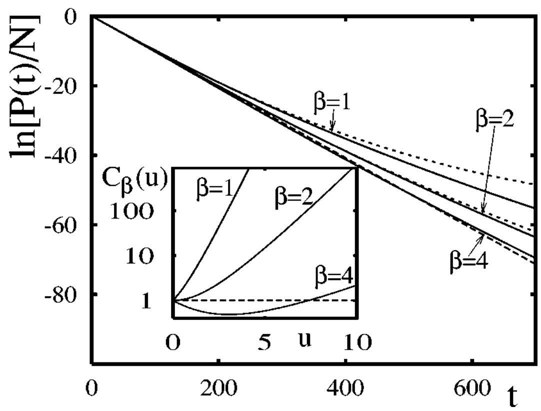

FIG. 1. Logarithm of $P(t)$ for the three symmetry classes with $E_{c,1}\approx 0.1$, $N=2000$ and all transmission eigenvalues $T_j=1$. The full lines are obtained from (14) for $\beta=2$ or the corresponding integrals for $\beta=1,4$. The dashed line shows the classical exponential decay and the two dotted lines for $\beta=1,2$ correspond to Eq. (2). For $\beta=4$, Eq. (2) coincides with the full line. The inset shows the functions $C_\beta(u)$ given in Eqs. (3)–(5).

$\xi=\beta D/2$. Here we concentrate on the case of a system size N being much smaller than ξ (i.e., $t_c\sim N^2/D\ll N=t_H$) and on time scales $t>t_c$.

We first present and discuss our main results before we outline some basic steps of the approach. We find that the first quantum corrections for $t_c<t\lesssim t_H^{2/3}t_c^{1/3}$ can be cast in the form

$$P(t)\propto e^{-E_{c,1}t}C_\beta\left(\frac{E_{c,2}t^2}{2N}\right),\quad E_{c,\nu}=\frac{g_\nu}{N},\tag{2}$$

where for the 0D limit ($k\approx N/2$ or for a chaotic cavity) we have introduced the ''generalized conductance'' moments by $g_\nu=\Sigma_j T_j^\nu$, $\nu=1,2,\ldots$. Here the transmission eigenvalues $0\le T_j\le 1$ describe the effective coupling strength of the cavity with the boundary. For the 1D limit, $g_1=g_2=\pi^2 D/2N$ is (up to a numerical factor) the classical conductance from the site 0 to the boundary. The universal functions $C_\beta(u)$ have the form (inset of Fig. 1)

$$C_1(u)=\int_0^1 dx\int_0^1 dy\,\frac{2x(1-x)}{(1-x^2+x^2y^2)^2}\times\exp[u(2x-(1-x^2+x^2y^2))]\tag{3}$$

$$=1+u+\frac{5}{6}u^2+\cdots,$$

$$C_2(u)=\sinh(u)/u=1+\frac{1}{6}u^2+\cdots\ ,\tag{4}$$

$$C_4(u)=C_1(-u/2)=1-\frac{1}{2}u+\frac{5}{24}u^2+\cdots\ .\tag{5}$$

For $\beta=1,2$ the quantum probability $P(t)$ is above its classical value $P_{\rm cl}(t)$. The criterion $\ln[P(t_q)/P_{\rm cl}(t_q)]=0.1$ (see [1]) to define the quantum relaxation time scale t_q leads to $t_q\approx 0.45\sqrt{N/E_{c,2}}$ for $\beta=1$ and $t_q\approx 1.24\sqrt{N/E_{c,2}}$ for $\beta=2$. The numerical factor for $\beta=1$ is indeed close to 0.38 found in [1]. We note that for $\beta=2$ the function $C_2(u)$ has only a quadratic correction for small u. The situation for $\beta=4$ is particularly intriguing because here the quantum probability is initially even *below* the classical value. The function $C_4(u)$ has at $u_{\rm min}\approx 3.03$ ($t_{q,\rm min}\approx 2.46\sqrt{N/E_{c,2}}$) its minimum value 0.488 and it crosses the classical value 1 again at $u_{\rm cr}\approx 7.36$ ($t_{q,\rm cr}\approx 3.84\sqrt{N/E_{c,2}}$). It seems that the linear term in the function $C_\beta(u)$ can be viewed as a weak-localization correction (antilocalization for $\beta=4$). In Fig. 1, we also show for the 0D case with $T_j=1$ the accurate result that is given by more complicated integrals (see below).

To derive these results, we have applied the supersymmetric technique [12,15], which has recently been generalized [14,19] to treat random phases instead of Gaussian disorder. Repeating the steps described in Ref. [19], we can express the Laplace transform $\tilde{P}(\omega)$ of $P(t)$ as a functional integral of the type

$$\tilde{P}(\omega)=\int \mathcal{D}Q\, f(Q(0))e^{-\mathcal{L}[Q]}.\tag{6}$$

Here the integration is done over a field of 8×8 supermatrices $Q(l)$, $-N/2\le l\le N/2$ with the nonlinear constraint $Q^2=1$ and particular symmetries for each universality class [12]. $f(Q(0))$ is a preexponential factor that depends only on the Q field at site 0. The action in Eq. (6) has the form

$$\mathcal{L}[Q]=\frac{d}{2}\,{\rm Str}_{8N}\,\ln(\hat{B}(\omega)+i\hat{Q}),\tag{7}$$

$$\hat{B}(\omega)=i\Lambda\,\frac{1-e^{i\omega/2}\hat{U}_0}{1+e^{i\omega/2}\hat{U}_0},\quad \hat{U}_0=\begin{pmatrix}U_0 & \\ & U_0^\dagger\end{pmatrix}.\tag{8}$$

The number $d=1$ (2) for $\beta=1,2$ ($\beta=4$) measures the spin degeneracy and the supertrace extends over an $8N$-dimensional super space. $\hat{Q}$ is an operator containing the $Q(l)$-fields in its diagonal blocks and U_0 is a matrix with elements $\langle l|e^{-iV(\theta)}|\tilde{l}\rangle\otimes\mathbb{1}_4$. The block structure in Eq. (8) refers to the grading for advanced and retarded Greens functions with the matrix Λ having the entries $+1$ (-1) in the upper (lower) diagonal block. As in [19], we expand the action in the limit of long wavelengths and long time scales, which gives $\mathcal{L}[Q]\approx\mathcal{L}_B[Q]+\mathcal{L}_{\rm 1D}[Q]$ where

$$\mathcal{L}_{\rm 1D}[Q]=-\frac{d}{32}\int_{-N/2}^{N/2}dl\ {\rm Str}(D(\partial_l Q)^2+4i\omega Q\Lambda)\tag{9}$$

is the standard one-dimensional σ model action. Here the supertrace Str without subscript acts on 8×8 supermatrices. The term $\mathcal{L}_B[Q]$ which was absent in [14,19] arises from the boundary absorption because the operator U_0 is *not* unitary due to the cutoff in l space. According to this we can write $\hat{B}(0)=B_1+i\Lambda B_2$ with Hermitian matrices B_1 and B_2. Note that B_2 does not vanish because U_0 is not unitary. The boundary part of the action is then determined by the eigenvalues $0\le T_j^{(0)}\le 1$ of the Hermitian matrix $\hat{T}^{(0)}=A^{-1/2}4B_2A^{-1/2}$ [with $A=B_1^2+(1+B_2)^2$]. These eigenvalues have the meaning of transparencies of coupling channels to the outside. Their precise distribution depends on microscopic details such as system size and the particular choice of the kick potential $V(\theta)$. The eigenvectors with nonvanishing $T_j^{(0)}$ have typically a support on the sites close to the boundary and the related boundary conductance $g^{(0)}=\Sigma_j T_j^{(0)}$ scales like the effective bandwidth of U_0: $g^{(0)}\sim\sqrt{D}$. We have verified this behavior by a numerical evaluation of $\hat{T}^{(0)}$ for the standard kicked rotator. Therefore we can write: $\mathcal{L}_B[Q]=L_B(\hat{T}^{(0)},Q(N/2))+L_B(\hat{T}^{(0)},Q(-N/2))$ with

$$L_B(\hat{T}^{(0)},Q)=\frac{d}{4}\sum_j \text{Str}\ln\left(1+\frac{1}{2}T_j^{(0)}\Delta Q\right) \tag{10}$$

and $\Delta Q=\frac{1}{2}(Q\Lambda+\Lambda Q)-1$. The sum runs over all nonvanishing eigenvalues associated to one boundary. We note that for the S-matrix approach of Refs. [15,20] exactly the same action is obtained where $T_j^{(0)}$ are the transmission eigenvalues of a tunnel barrier which couples a mesoscopic sample to an ideal quantum wire [20].

The functional integral (6) corresponds to a path integral that can be evaluated by solving a diffusion equation in Q space [21,12,22]. Therefore we rewrite Eq. (6) as

$$\widetilde{P}(\omega)=\int dQ\, f(Q)F^2(Q,N/2), \tag{11}$$

where the function $F(Q,l)$ is determined by the partial differential equation [21,12,22]

$$\partial_l F(Q,l)=\left(\frac{2}{\xi}\Delta_Q+i\frac{d}{8}\omega\,\text{Str}(Q\Lambda)\right)F(Q,l) \tag{12}$$

and the initial condition $F(Q,0)=\exp[-L_B(\hat{T}^{(0)},Q)]$. Here Δ_Q denotes the Laplace operator in Q space (with the precise notations of Ref. [23]). The general solution of Eq. (12) for arbitrary frequencies is an involved mathematical problem. First, we consider the solution $F_0(Q,l)$ for the case $\omega=0$. For this, we note that $\exp[-L_B(\hat{T},Q)]$ as a function of T_j and Q exactly coincides with the generating function (2.3) of Ref. [23], which was used to prove the equivalence of the σ model [21,12,22] and Fokker-Planck approach [24–26] for quasi one-dimensional disordered wires. According to the argumentation presented in [23], $F_0(Q,l)$ is exactly given by

$$F_0(Q,l)=\int d\hat{T}\,p(\hat{T},l)\exp[-L_B(\hat{T},Q)] \tag{13}$$

where $p(\hat{T},l)$ is a probability distribution of transmission eigenvalues T_j, which fulfills a certain Fokker-Planck equation (known as DMPK-equation due to Dorokhov [24], and Mello, Pereyra, Kumar [25]) with the initial condition $p(\hat{T},0)=\delta(\hat{T}-\hat{T}^{(0)})$. $p(\hat{T},l)$ describes the statistical transport properties of a quasi-one-dimensional disordered wire in series with a tunnel barrier with transparencies $T_j^{(0)}$. At first sight Eq. (13) seems to be more complicated due to the increased number of integrations. However, in the metallic limit, we can expand Eq. (10) in powers of ΔQ with the self-averaging transmission moments $g_1,g_2,g_3,\ldots$ as prefactors. Their ''quantum'' fluctuations are of order unity and have only an effect for $t\gtrsim t_H$. Therefore, we can replace g_ν by their average values and omit the T average. These g_ν averages are in the classical limit determined by a set of differential equations that can be derived from the DMPK equation [26]. To determine $F(Q,l)$ for $\omega\neq 0$ we use the expression for $F_0(Q,l)$ as an ansatz where the g_ν are now parameters to be determined as a function of ω. The ω term only modifies the equation for g_1 giving $g_1'(l)=-(2/D)g_1^2-i\omega$ and $g_2'(l)=(4/D)(g_1^2-2g_1g_2)$. Omiting the details, we mention that the explicit solutions determine $F(Q,l)$ and thus provide a closed expression for $\widetilde{P}(\omega)$ as one Q integral (11). Using the standard parametrizations for Q introduced by Efetov [12], we can express Eq. (11) as an integral over two ($\beta=2$) or three ($\beta=1,4$) radial parameters. We can perform the integrations for ω (from the Fourier transform) and for the effective variable $s=\text{Str}(\Delta Q)$ in a saddle point approximation, which is justified for $t\gg t_c$. Keeping the first two terms with g_1 and g_2 in $F(Q,l)$ we obtain our main result (2)–(5) for the 1D case. The situation for the 0d case is much easier, here we can simply insert the given ''boundary'' transmission eigenvalues and perform the ω integration. For lack of space, we only state the result for $\beta=2$

$$P(t)\approx\frac{1}{t}\int_0^{\min(1,t/N)}dx\left(1+2\frac{t}{N}-2x\right)e^{-L(x)}, \tag{14}$$

$$L(x)=\sum_j \ln\left(\frac{1+(t/N-x)T_j}{1-xT_j}\right). \tag{15}$$

The corresponding expressions for $\beta=1,4$ have a similar structure with two integrations. The curves shown in Fig. 1 were obtained from a numerical evaluation of these integrals. They also lead to our principal result (2)–(5) if we expand the logarithm in Eq. (15) up to second order in T. The expansion parameter here is, in principle, $t/N\sim t/t_H\ll 1$. However, one can estimate that the third order term gives a contribution $\propto t^3/(t_c t_H^2)$, which has to be smaller than unity because of the exponential in Eq. (14). Of course the same criterion holds for the 1D case if we restrict ourselves to the first two moments g_1 and g_2.

In summary, we have found that for open chaotic systems the first quantum corrections to the classical relaxation process appear at a quantum relaxation time scale $t_q\sim\sqrt{t_c t_H}$ with different effects for each universality class (Fig. 1). This scale is determined by the second moment of transmissions eigenvalues T_j describing the effective coupling strength of the initial site with the boundary. It would be very interesting to relate this finding more clearly to the physical mechanism suggested in Ref. [1], according to which t_q is the time scale at which the quantum discreteness of the complex eigenvalues $\exp(iE_j-\Gamma_j/2)$ of the nonunitary time evolution operator S [2] can be resolved. We emphasize that in view of the universal σ model formulation our results apply not only to the kicked rotator model (1) but also to chaotic cavities (corresponding to the zero-dimensional random matrix limit) and to quasi-one-dimensional disordered wires. In this case one should consider the time evolution of a wave packet of plane waves in an energy interval of size $\hbar/\tau$, where τ is the elastic scattering time. The typical extension of a such a wave packet is just the mean free path, which is in any case the smallest length scale that can be resolved by the standard σ model [12].

Due to the almost identical σ model action it is important to understand the relation of our results with the approach of Ref. [10], where mainly the limit $t > t_H$ was considered. A recent careful analysis [27] of the saddle point approach pioneered in Ref. [10] indeed gives for the regime $t_q \ll t \ll t_H$ the behavior $\ln P(t) \approx -(t/t_c)[1 - t/(\beta g t_c)]$ confirming Eqs. (2)–(5) for $u \gg 1$. Furthermore, for $t > t_H$ we can state that the log-normal behavior found in [9–11] should also apply to the *average* decay rate for the kicked rotator model. However, for very long time scales one should also focus on the distribution of the decay function because for a *given* sample the decay is then again exponential with a decay rate given by the minimal Γ_j [1].

Concerning the zero-dimensional limit, our result (2), (3) for $\beta = 1$ is, in principle, also contained in the exact integral expressions of [13]. However, since the corresponding limit was not worked out there the time scale t_q remained undetected. We emphasize that here the T_j are given model parameters and $E_{c,2}$ might parametrically be smaller than $E_{c,1}$ if all $T_j \ll 1$. We mention that very recently Savin and Sokolov [28] independently also found the time scale t_q in the frame work of the supersymmetric approach. Their results, which apply for the 0D case with unitary symmetry, completely agree with our findings (2) and (14).

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